Steady Heat Conduction in Cartesian Coordinates and a Library of Green's Functions

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Motivation

Verification of fully-numeric codes

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Geometry: Parallelepiped
Outline

- Temperature problem, Cartesian domains
- Green’s function solution
- Green’s function in 1D, 2D and 3D
- Web-based Library of Green’s Functions
- Summary
Temperature Problem

\[ \nabla^2 T = -\frac{g}{k} \quad \text{in a finite domain} \ \mathcal{R} \]

\[ k_i \frac{\partial T}{\partial n_i} + h_i T = f_i \quad \text{on the i^{th} boundary} \]

Domain \( \mathcal{R} \) includes the slab, rectangle, and parallelepiped.

The boundary condition represents one of three types:
Type 1. \( k_i=0, \ h_i=1, \) and \( f_i \) a specified temperature;
Type 2. \( k_i=k, \ h_i=0, \) and \( f_i \) a specified heat flux \([\text{W/m}^2]\);
Type 3. \( k_i=k \) and \( h_i \) a heat transfer coefficient \([\text{W/m}^2/\circ K] \).
What is a Green's Function?

Green's function (GF) is the response of a body (with homogeneous boundary conditions) to a concentrated energy source. The GF depends on the differential equation, the body shape, and the type of boundary conditions present.

Given the GF for a geometry, any temperature problem can be solved by integration.

Green's functions are named in honor of English mathematician George Green (1793-1841).
Green’s function solution

\[ T(r) = \int \frac{g(r')}{k} G(r \mid r') \, dv' \quad \text{(for volume energy generation)} \]

\[ + \sum_i \int_{s_i} \frac{f_i}{k} G(r \mid r'_{i}) \, ds'_{i} \quad \text{(for b. c. of type 2 and 3)} \]

\[ - \sum_i \int_{s_i} f_i \frac{\partial G(r \mid r'_{i})}{\partial n_{i}} \, ds'_{i} \quad \text{(for b. c. of type 1 only)} \]

Green’s function \( G \) is the response at location \( r \) to an infinitesimal heat source located at coordinate \( r' \).
Green’s function for 1D Slab

\[
\frac{d^2 G}{dy^2} = -\delta(y - y'); \quad 0 < y < W
\]

\[
k_i \frac{dG}{dn_i} + h_i G = 0; \quad i = 1 \text{ or } 2
\]

Boundary conditions are homogeneous, and of the same type \((1, 2, \text{ or } 3)\) as the temperature problem. There are \(3^2 = 9\) combinations of boundary types for the 1D slab.
1D Example

G=0 at y=0 and at y=W.

Y11 case. Two forms:

Series.

\[ G(y, y') = \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{\sin(\gamma_n y) \sin(\gamma_n y')}{W/2} \]

Polynomial.

\[ G(y, y') = \begin{cases} 
  y(1 - y'/W); & y < y' \\
  y'(1 - y/W); & y > y'
\end{cases} \]
Y11 case, continued

Plot of $G(y,y')$ versus $y$ for several $y'$ values.

Y11 Geometry.
GF for the 2D Rectangle

\[
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x - x')\delta(y - y'); \quad 0 < x < L; \quad 0 < y < W
\]

\[
k_i \frac{\partial G}{\partial n_i} + h_i G = 0 \quad \text{for faces } i = 1, 2, 3, 4
\]

• Here G is dimensionless.
• There are \(3^4 = 81\) different combinations of boundary conditions (different GF) in the rectangle.
2D Example

Case X21Y11. $G=0$ at edges, except insulated at $x=0$.

Double sum form:

$$G(x, y \mid x', y') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\lambda_m x') \cos(\lambda_m x) \sin(\gamma_n y) \sin(\gamma_n y')}{(L/2)(W/2)} \frac{\sin^2(\gamma_n y')}{(\gamma_n^2 + \lambda_m^2)}$$

where

$$\gamma_n = n\pi / W$$

$$\lambda_m = (m - 1/2)\pi / L$$
2D Example, case X21Y11

Single sum form:

\[ G(x, y | x', y') = \sum_{n=1}^{\infty} \frac{\sin(\gamma_n y)\sin(\gamma_n y')}{W/2} P_n(x, x') \]

where kernel function \( P_n \) for this case is:

\[
P_n(x, x') = \left\{ -\exp[-\gamma_n(2L - |x - x'|)] - \exp[-\gamma_n(2L - x - x')] \right. \\
+ \exp[-\gamma_n |x - x'|] + \exp[-\gamma_n(x + x')] \\
\left. \div \{2\gamma_n[1 + \exp(-2\gamma_n L)]\} \right\}
\]
Case X21Y11 heated at (0.4,0.4)
GF for the 3D Parallelepiped

\[
\begin{align*}
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} &= -\delta(x-x')\delta(y-y')\delta(z-z') \\
0 < x < L; 0 < y < W; 0 < z < H \\
k_i \frac{\partial G}{\partial n_i} + h_i G &= 0 \text{ for faces } i = 1, 2, ..., 6
\end{align*}
\]

There are $3^6 = 729$ combinations of boundary types.
3D Example
Case X21Y11Z12

Triple sum form:

\[
G = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\cos(\lambda_m x) \cos(\lambda_m x') \sin(\gamma_n y) \sin(\gamma_n y')}{(L/2)(W/2)(H/2)} \times \frac{\sin(\eta_p z) \sin(\eta_p z')}{(\lambda_m^2 + \gamma_n^2 + \eta_p^2)}
\]
3D Example, X21Y11Z12
Alternate double-sum forms:

\[
G = \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(\gamma_n y') \sin(\gamma_n y)}{(W/2)(H/2)} \frac{\sin(\eta_p z) \sin(\eta_p z')}{(\gamma_n^2 + \eta_p^2)} P_{np}(x, x')
\]

\[
G = \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos(\lambda_m x') \cos(\lambda_m x)}{(L/2)(H/2)} \frac{\sin(\eta_p z) \sin(\eta_p z')}{(\lambda_m^2 + \eta_p^2)} P_{mp}(y, y')
\]

\[
G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos(\lambda_m x') \cos(\lambda_m x)}{(L/2)(W/2)} \frac{\sin(\gamma_n y) \sin(\gamma_n y')}{(\lambda_m^2 + \gamma_n^2)} P_{mn}(x, x')
\]
Web Publication: Promise

- Material can be presented in multiple digital formats, may be cut and pasted into other digital documents.
- Immediate world-wide distribution.
- Retain control of content, easily updated.
- Hyperlinks to related sites.
Web Publication: Pitfalls

- No editorial support, no royalties.
- Unclear copyright protection.
- Continued operating costs (service provider, computer maintenance, etc.)
- Little academic reward; doesn’t “count” as a publication.
NIST Digital Library of Mathematical Functions

- Web-based revision of handbook by Abramowitz and Stegun (1964).
- Emphasis on text, graphics with few colors, photos used sparingly.
- Navigational tools on every page.
- No proprietary file formats (HTML only).
- Source code developed in AMS-TeX.
Green’s Function Library

- Source code is LaTeX, converted to HTML with shareware code latex2html run on a Linux PC
- GF are organized by equation, coordinate system, body shape, and type of boundary conditions
- Each GF also has an identifying number
Contents of the GF Library

• Heat Equation. Transient Heat Conduction
  Rectangular Coordinates. Transient 1-D
  Cylindrical Coordinates. Transient 1-D
  Radial-Spherical Coordinates. Transient 1-D

• Laplace Equation. Steady Heat Conduction
  Rectangular Coordinates. Steady 1-D
  Rectangular Coordinates. Finite Bodies, Steady.
  Cylindrical Coordinates. Steady 1-D
  Radial-Spherical Coordinates. Steady 1-D

• Helmholtz Equation. Steady with Side Losses
  Rectangular Coordinates. Steady 1-D
Green's Function Library

The purpose of the Green's Function (GF) Library is to organize solutions of linear differential equations and to make them accessible on the World Wide Web.


Starting points.

- Contents of the GF Library.
- Organization of the GF Library---GF Numbering System.
- Search for Green's Functions.
Plate, steady 1-D.

X11 Plate, \( G=0 \) (Dirichlet) at \( x=0 \) and \( x=L \).

\[
G_{X11}(x \mid x') = \begin{cases} 
  \frac{x(1-s')}{L} & \text{for } s < s' \\
  \frac{s'(1-x)}{L} & \text{for } s > s'
\end{cases}
\]

X12 Plate, \( G=0 \) (Dirichlet) at \( x=0 \) and \( \frac{\partial G}{\partial x} = 0 \) (Neumann) at \( x=L \).

\[
G_{X12}(x \mid x') = \begin{cases} 
  s & \text{for } s < s' \\
  s' & \text{for } s > s'
\end{cases}
\]

X13 Plate, \( G=0 \) (Dirichlet) at \( x=0 \) and \( k \frac{\partial G}{\partial x} + h_2 G = 0 \) (Neumann) at \( x=L \). Note \( B_2 = h_2 L / k \).

\[
G_{X13}(x \mid x') = \begin{cases} 
  \frac{x[1-B_2(x'/L)]/(1+B_2)}{(1+B_2)} & \text{for } s < s' \\
  \frac{x'[1-B_2(x'/L)]/(1+B_2)}{(1+B_2)} & \text{for } s > s'
\end{cases}
\]
Solid cylinder transient 1-D.

R01 Solid cylinder $0 < r < b$, with $G = 0$ (Dirichlet) at $r = b$.

$$G_{R01}(r, t | r', \tau) = \frac{1}{\pi b^2} \sum_{m=1}^{\infty} \exp \left[ -\beta_m^2 \alpha (t - \tau) / b^2 \right]$$

$$\times \frac{J_0(\beta_m r / b) J_0(\beta_m r' / b)}{[J_1(\beta_m)]^2}$$

with eigenvalues given by $J_0(\beta_m) = 0$. 
Summary

• GF in slabs, rectangle, and parallelepiped for 3 types of boundary conditions

• These GF have components in common: 9 eigenfunctions and 18 kernel functions

• Alternate forms of each GF allow efficient numerical evaluation
Summary, continued.

Web Publishing: wide dissemination, local control, updatable; continuing expense, little academic reward.

Green’s Function Library: source code developed in LaTeX (runs on any computer) and converted to HTML with latex2html (runs on Linux).
Work in progress: Dynamic Math

- Currently GF web page is static, book-like
- Temperature solutions are too numerous for pre-determined display
- Working to create and display temperature solutions on demand, in response to user input.
- Code with open standards Perl, latex2html
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