SURVEY OF GREEN'S FUNCTION RESEARCH RELATED TO TRANSIENT HEAT CONDUCTION

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MOTIVATION

UNIQUE OPPORTUNIES NOW

Exploding knowledge
“Infinite” computer memory
World-wide internet connectivity

APPLICATIONS

Education
Retention/availability of unique contributions
Verification

THREE-DIMENSIONAL HEAT CONDUCTION EQUATION IN CARTESIAN COORDINATES

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+g=C\,\frac{\partial T}{\partial t}$$

Domain: 0 < *x* < *L*, 0 < *y* < *W*, 0 < *z* < *H*

T **= Temperature**
k **= Thermal conductivity**
g **= Volumetric Energy Generation**
C **= Volumetric Heat Capacity**
α **= *k/C* = Thermal Diffusivity**

Boundary Conditions:

$T(0,y,z,t) = T_0$, Prescribed T, 1st kind

$$k\frac{\partial T(L,y,z,t)}{\partial x} = q_0$$ **, Prescribed heat flux, 2nd kind**

$$-k\frac{\partial T(x,0,z,t)}{\partial y} = h\left(T_\infty - T(x,0,z,t)\right)$$ **, Convective, 3rd kind**

If no physical boundary, such as for $x \rightarrow \infty$ or r in radial coordinates $\rightarrow$, 0th kind.

GREEN'S FUNCTION SOLUTION EQUATION

For homogeneous, *T*-independent properties:

$$T(\mathrm{r},t) = T_{in}(\mathrm{r},t) + T_g(\mathrm{r},t) + T_{b.c.}(\mathrm{r},t)$$

v

where the initial condition term is

$$T_{in}(\mathrm{r},t) = \int_V G(\mathrm{r},t \,|\, \mathrm{r}',0)\, F(\mathrm{r}')\, dv'$$

For volumetric energy generation,

$$T_g(\mathrm{r},t) = \frac{\alpha}{k} \int_{\tau=0}^{t} \int_V G(\mathrm{r},t \,|\, \mathrm{r}',\tau)\, g(\mathrm{r}',\tau)\, dv'\, d\tau$$

For the nonhomogeneous boundary conditions

$$T_{b.c.}(\mathrm{r},t)=\frac{\alpha}{k}\int_{\tau=0}^{t}\sum_{i=1}^{s}\int_{A_i} G(\mathrm{r},t|\mathrm{r}_i',\tau)\, f_i(\mathrm{r}_i',\tau)ds_i\,'d\tau$$

$$+\alpha\int_{\tau=0}^{t}\sum_{j=1}^{s}\int_{A_j}\left(-\frac{\partial G(\mathrm{r},t|\mathrm{r}_j',\tau)}{\partial n'}\bigg|_{\mathrm{r}'=\mathrm{r}_j'}\right)f_j(\mathrm{r}_j',\tau)ds_j\,'d\tau$$

where the first line is for b.c. of 2nd & 3rd kinds and second line is for b.c. of 1st kind.

For homogeneous rectangle or parallelepiped, G(r,*t*|r',τ) can be written as a product of 1D GFs. For 3D case,

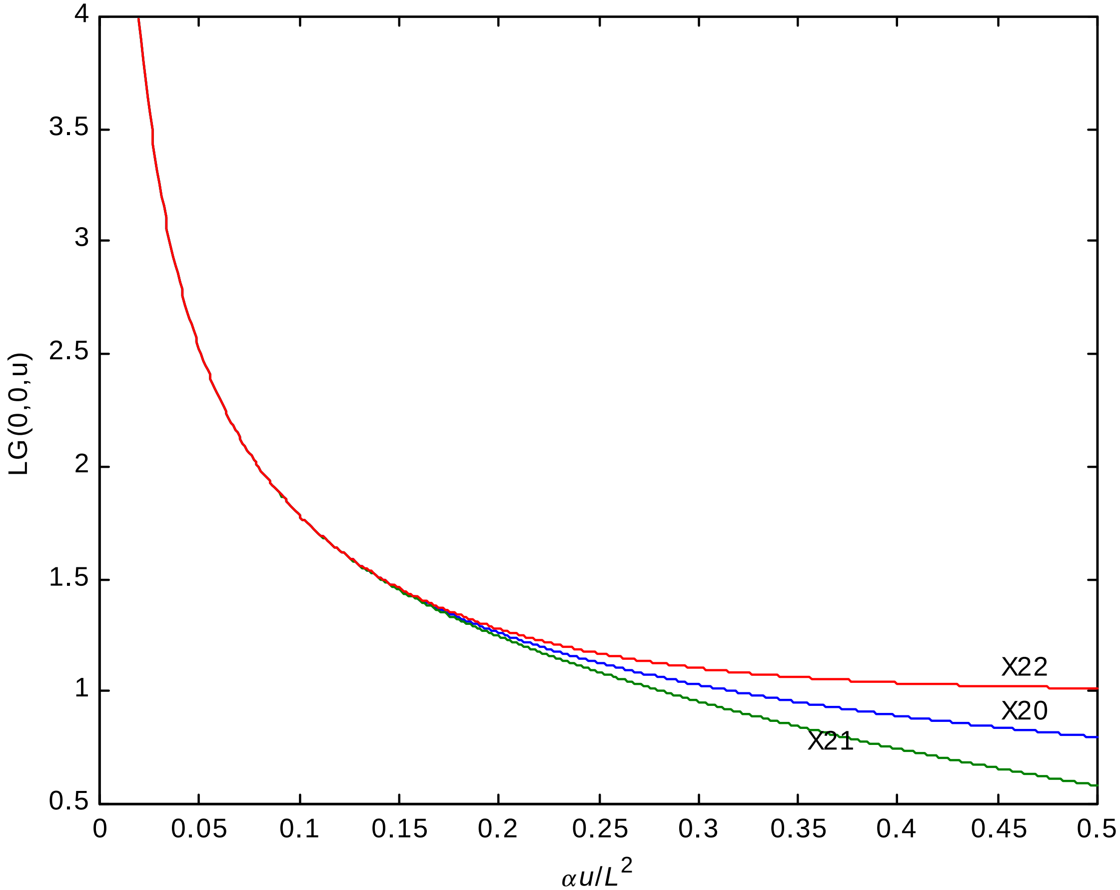
$$G(\mathrm{r},t|\mathrm{r}',\tau)=G_X(x,t|x',\tau)G_Y(y,t|y',\tau)G_Z(z,t|z',\tau)$$

Consider a boundary condition of 2nd kind at *x* = 0 (a constant heat flux, q_0) and boundary conditions of the 1st kind at all the other 5 boundaries. Then

$$G(\mathrm{r},t|\mathrm{r}',\tau)=G_{X21}(x,t|x',\tau)G_{Y11}(y,t|y',\tau)G_{Z11}(z,t|z',\tau)$$

$$T_{b.c.}(x,y,z,t)=$$

$$\frac{\alpha}{k}q_0\int_{\tau=0}^{t}G_{X21}(x,t|0,\tau)\int_{y'=0}^{W}G_{Y11}(y,t|y',\tau)dy'\int_{z'=0}^{H}G_{Z11}(z,t|z',\tau)dz'd\tau$$



LG*$_{X20}$(0,0,*u*), *LG*$_{X21}$(0,0,*u*), and *LG*$_{X22}$(0,0,*u*) vs. dimensionless time; *u* ≡ *t* - *τ

$\alpha u/L^2$	$LG_{X20}(0,0,u)$	$LG_{X21}(0,0,u)$	$LG_{X22}(0,0,u)$
0.040	**2.820947918**	**2.820947918**	**2.820947918**
0.050	**2.523132522**	**2.523132512**	**2.523132532**
0.060	**2.303294330**	**2.303294064**	**2.303294596**
0.070	**2.132436186**	**2.132433521**	**2.132438851**

$$LG_{X20}(0,0,u) = (\pi\alpha u / L^2)^{-1/2}$$

$$LG_{X21}(0,0,u) = 2\sum_{m=1}^{\infty} e^{-\left((2m-1)\frac{\pi}{2}\right)^2 \frac{\alpha u}{L^2}}$$

$$LG_{X22}(0,0,u) = 1 + 2\sum_{m=1}^{\infty} e^{-(m\pi)^2 \frac{\alpha u}{L^2}}$$

OBSERVATIONS

1. **Large part of GF at short (i.e., recent) times.**

2. **Accurate & simple approximation for short times.**

3. **For long time GF, want $\exp[-(m_{max}\pi)^2\alpha u/L^2]$ to be small. Note $\exp(-2\pi^2) \approx$ 3E-9. Then $m_{max} = L(2/\alpha u)^{1/2}$**

4. **Fewer terms in long time GF for small *L* and large *u*.**

WAYS TO IMPROVE EFFICIENCY AND ACCURACY

A. **Use short and long time GF. Time partitioning.**
B. **Use maximum *t* possible in long time GF.**
C. **For long time GF and small t, use artificially small L. Spatial partitioning.**

NUMBERING SYSTEM

MOTIVATION-Many geometries and boundary conditions

EXAMPLE: Temperature b.c. = 1^{st} kind

Heat flux b.c. = 2^{nd} kind

Convective b.c. = 3^{rd} kind

No physical boundary= 0^{th} kind

In heat conduction, one b.c. at each boundary.

***X* for *x*-direction, *Y* for *y*-direction, *Z* for *z*-direction**

XIJ* is for plate with *I*th b.c. at *x* = 0, and *J*th b.c. at *L

Suppose *T* given at *x* = 0 & convection b.c. at *L*: *X*13

Different possibilities in *x*-coordinates:

***X*00**
X10 X11 X12 X13
X20 X21 X22 X23
X30 X31 X32 X33

A Green's function can be given for each of these.

Except for of 0th kind, we give TWO forms of each.

They are complementary in that one is more efficient than the other in different time domains.

They can be used to provide internal verification.

EXAMPLE *X*11 GF

Form best for small t - τ,

$$G_{X11}(x,t|x',\tau)=\frac{1}{\sqrt{4\pi\alpha(t-\tau)}}\sum_{n=-\infty}^{\infty}\left[e^{-\frac{(2nL+x-x')^2}{4\alpha(t-\tau)}}-e^{-\frac{(2nL+x+x')^2}{4\alpha(t-\tau)}}\right]$$

Form best for large t - τ,

$$G_{X11}(x,t|x',\tau)=\frac{2}{L}\sum_{m=1}^{\infty}e^{-\beta_m\frac{\alpha(t-\tau)}{L^2}}\sin\left(\beta_m\frac{x}{L}\right)\sin\left(\beta_m\frac{x'}{L}\right) \quad \beta_m=m\pi$$

By selecting $\alpha(t-\tau)/L^2\approx0.05$, only a few terms needed.
Related to our method of "time partitioning"

Also some functions of GFs are convenient to have:

$$-\frac{\partial G_{X11}}{\partial n'}\bigg|_{n'=0}, \quad -\frac{\partial^2 G_{X11}}{\partial x \partial n'}\bigg|_{n'=0}, \quad \int_{x'=0}^{L} G_{X11} dx', \quad \int_{x'=0}^{L} \frac{\partial G_{X11}}{\partial x} dx'$$

For 3D problems in Cartesian coordinates, $G = G_X\, G_Y\, G_Z$

CYLINDRICAL RADIAL & RADIAL/ANGULAR

Many of these are also tabulated in our book, "Heat Conduction Using Green's Functions" by J.V. Beck, K.J. Cole, A. Haji-Sheikh and B. Litkouhi

In the book, radial spherical Green's functions are given.

CONDUCTION WITH SOLID BODY FLOW

Consider the differential equation:

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+g=C\left[\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}+W\frac{\partial T}{\partial z}\right]$$

where *U, V* and *W* are constants and velocities in the *x, y* and z directions, resp.
The same boundary conditions as used above are used.

Flow eq. can be transformed to heat conduction using:

$$T(x,y,z,t)=W^*(x,y,z,t)\exp\left[\frac{Ux}{2\alpha}-\frac{U^2t}{4\alpha}+\frac{Vy}{2\alpha}-\frac{V^2t}{4\alpha}+\frac{Wz}{2\alpha}-\frac{W^2t}{4\alpha}\right]$$

The boundary and initial conditions change also.

The b.c. of first kind remains of the 1st kind.

The b.c. of second kind is changed to the 3rd kind.

The b.c. of third kind remains the 3rd kind.

EXAMPLES OF GF WITH FLOW

General GF equation for long-time type for flow in the *x*-direction is

$$G_{XUIJ}(x,t\,|\,x',\tau)=\frac{X_0(x)X_0(x')}{N_0}e^{\frac{Pe_x}{2}\frac{x-x'}{L}+\left(\beta_0^2-\left(\frac{Pe_x}{2}\right)^2\right)\frac{\alpha(t-\tau)}{L^2}}+e^{\frac{Pe_x}{2}\frac{x-x'}{L}}\sum_{m=1}^{\infty}e^{-R_m^2\frac{\alpha(t-\tau)}{L^2}}\frac{X_m(x)X_m(x')}{N_m}$$

where

$$Pe_x \equiv \frac{UL}{\alpha}, \quad R_m^2 \equiv \beta_m^2 + \left(\frac{Pe_x}{2}\right)^2$$

The above equation is for *T*, not transformed variable W.

XU11

Short time form

$$G_{XU11}^S(x,t|x',\tau) \approx$$

$$e^{\frac{Pe_x}{2}\frac{x-x'}{L}} e^{-\left(\frac{Pe_x}{2}\right)^2 \frac{\alpha(t-\tau)}{L^2}} \begin{bmatrix} K(x-x') - K(x+x') - \\ K(2L-x-x') + K(2L-x+x') + K(2L+x-x') \end{bmatrix}$$

where

$$K(z) \equiv \frac{1}{(4\pi\alpha u)^{1/2}} e^{-\frac{z^2}{4\alpha u}}, \quad u \equiv t-\tau$$

$$H_0(z) \equiv e^{-\frac{Pe_x z}{2L}+\frac{Pe_x^2}{4}\frac{\alpha u}{L^2}} \operatorname{erfc}\left(\frac{z}{L}\left[\frac{4\alpha u}{L^2}\right]^{-1/2} - \frac{Pe_x}{2}\left[\frac{\alpha u}{L^2}\right]^{1/2} \right)$$

We need derivatives with respect to *x'*; one is

$$-\frac{\partial G^S_{XU11}(x,t\,|\,0,\tau)}{\partial n'} \approx$$

$$\frac{1}{\alpha u} e^{\frac{Pe_x}{2}\frac{x}{L}} e^{-\left(\frac{Pe_x}{2}\right)^2 \frac{\alpha u}{L^2}} \left[xK(x) - (2L-x)K(2L-x) + (2L+x)K(2L+x) \right]$$

Long time form

$$G_{XU11}(x,t|x',\tau)=\frac{2}{L}e^{\frac{Pe_x}{2}\frac{x-x'}{L}}\sum_{m=1}^{\infty}e^{-R_m^2\frac{\alpha(t-\tau)}{L^2}}\sin\left(m\pi\frac{x}{L}\right)\sin\left(m\pi\frac{x'}{L}\right)$$

XU22

Short time form

$$G_{XU22}^{S}(x,t\,|\,x',\tau)\approx$$

$$e^{\frac{Pe_x}{2}\frac{x-x'}{L}}e^{-\left(\frac{Pe_x}{2}\right)^2\frac{\alpha(t-\tau)}{L^2}}\left[\begin{array}{l}K(x-x')+K(x+x')+K(2L-x-x')+\\K(2L-x+x')+K(2L+x-x')+K(2L+x+x')+\\\frac{1}{L}\frac{Pe_x}{2}[H_0(x+x')-H_L(2L-x-x')]\end{array}\right]$$

Long time form

$$G_{XU22}(x,t|x',\tau) = \frac{Pe_x}{\mathrm{L}} \frac{e^{-Pe_x \frac{x'}{L}}}{1-e^{-Pe_x}} +$$

$$\frac{2}{L} e^{\frac{Pe_x}{2}\frac{x-x'}{L}} \sum_{m=1}^{\infty} e^{-R_m^2 \frac{\alpha(t-\tau)}{L^2}} \frac{\left[m\pi\cos\left(m\pi\frac{x}{L}\right) - \frac{Pe_x}{2}\sin\left(m\pi\frac{x}{L}\right)\right]\left[m\pi\cos\left(m\pi\frac{x'}{L}\right) - \frac{Pe_x}{2}\sin\left(m\pi\frac{x'}{L}\right)\right]}{R_m^2}$$

TWO LAYER PARALLELEPIPED

A. Haji-Sheikh, David Yen

For 0 < *x* < *A*, 0 < *y* < *B*, 0 < *z* < *D*

$$k_1\left[\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_1}{\partial y^2}+\frac{\partial^2 T_1}{\partial z^2}\right]+g_1=C_1\frac{\partial T_1}{\partial t}$$

For 0 < *x* < *A*, *B* < *y* < *C*, 0 < *z* < *D*

$$k_2\left[\frac{\partial^2 T_2}{\partial x^2}+\frac{\partial^2 T_2}{\partial y^2}+\frac{\partial^2 T_2}{\partial z^2}\right]+g_2=C_2\frac{\partial T_2}{\partial t}$$

BOUNDARY CONDITIONS

At $x = 0$ & A, $z = 0$ & C: 1st and 2nd kinds

At $y = 0$ & C: 1st, 2nd, and 3rd kinds

INTERFACE RESISTANCE

EIGENVALUES (Only long time form available)

Nine different conditions, (*X*11, *X*12, etc.)

Some are imaginary; some are close to each other.

$$G_{ij}(x,y,z,t|x',y',z',\tau)=$$

$$\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{C_j X_m(x)X_m(x')Z_n(z)Z_n(z')Y_{i,mnp}(y)Y_{i,mnp}(y')}{N_{x,m}N_{z,n}N_{y,mnp}}e^{-\lambda_{mnp}^2(t-\tau)}$$

i **and** ***j*** **= 1 and 2.**

(Not independent product of the 3 components)

Reference: A. Haji-Sheikh and J.V. Beck, Int. J. of Heat and Mass Transfer, Vol. 45, (2002) p. 1865-1877)

IMPLEMENTATION TO CALCULATE *T* AND HEAT FLUX

1. **Time partitioning: Use both short and long time GF in same problem. Needs numerical integration.**

2. **Spatial partitioning: Use small part volume in one temporal/spatial domain, then a larger one, etc. Avoids need of numerical integration over time.**

3. **Unsteady surface element method: For connecting two basic and different geometries**

COMPUTER PROGRAMS

COND3D. Parallelepiped, homogeneous body. B.C. of the 1st, 2nd and 3rd kinds on all six boundaries.

Nonzero initial temperature. Vol. Energy Generation.

Uniform conditions over a surface or volume, zero or constant. Many possible cases.

Highly accurate, to 1 part in 10^{10} of maximum value.

Internal verification. Should get the "same" as the partition time is varied.

EXAMPLE. Parallelepiped, L = 0.1m, W = 0.05m, H = 0.025m at x = 0.075m, y = 0.0125m, z = 0m

$x = 0$: q = 3500 W/m^2; $x = L$: T = 1000°C

y = 0: q = 0 W/m^2; $y = W$: T_∞ = 25°C, Ht. Trans. Coef 60 W/m^2•C

z = 0: T_∞ = 50°C, Ht. Trans. Coef 10 W/m^2•C; z = H: T_∞ = 40°C, Ht. Trans. Coef 5 W/m^2•C

k = 0.4 W/m•C; C = 3000000 J/m^3•C; t = 1000s

T_0 = 100°C; Vol. Energy Gen. = 135300 W/m^3

COND3D RESULTS. Same to 10 sign. figures for t_p = 0.025, 0.05. This provides VERIFICATION.

Temperature	*x*-heat flux	*y*-heat flux	*z*-heat flux
208.7524786	-4261.888204	60.6443709	-1587.524786

These values agree with a similar, but not identical, program to all 10 sign. digits, except one with a 5 instead of 4 in the last digit.

CONSIDER STEADY STATE FOR SAME PROBLEM, time = 10,000,000 s

COND3D (TRANSIENT PROGRAM)
TEMPERATURE = 447.994631779662
HEAT FLUX (X) = -4146.97213360952
HEAT FLUX (Y) = 775.803316289508
HEAT FLUX (Z) = -3979.94631779662

VERIFSS (KEVIN COLE STEADY STATE)
The temperature is 447.99463177966
The flux is -4146.9721336095 775.80331628950 -3979.9463177966

Agree to within about 13 or 14 digits. Completely different programs.

SUMMARY

- **Digital databases now possible**
- **Verification.**
  Internal verification
  Extreme accuracy possible
- **Green's functions in heat conduction given**
- **Prototype of database given**