Steady Heat Conduction in Cartesian Coordinates and a Library of Green's Functions

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Motivation

Verification of fully-numeric codes

Sponsor: Sandia National Laboratory

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Geometry: Parallelepiped

Outline

- Temperature problem, Cartesian domains
- Green’s function solution
- Green’s function in 1D, 2D and 3D
- Web-based Library of Green’s Functions
- Summary

Temperature Problem

$$\nabla^2 T = -\frac{g}{k} \quad \text{in a finite domain } \mathbf{R}$$

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i \quad \text{on the } i^{th} \text{ boundary}$$

Domain **R** includes the slab, rectangle, and parallelepiped.

The boundary condition represents one of three types :
Type 1. k_i=0, h_i=1, and f_i a specified temperature;
Type 2. k_i=k, h_i=0, and f_i a specified heat flux [W/m^2];
Type 3. k_i=k and h_i a heat transfer coefficient [$W/m^2/{}^\circ K$].

What is a Green's Function?

Green's function (GF) is the response of a body (with homogeneous boundary conditions) to a concentrated energy source. The GF depends on the differential equation, the body shape, and the *type* of boundary conditions present.

Given the GF for a geometry, *any* temperature problem can be solved by integration.

Green's functions are named in honor of English mathematician George Green (1793-1841).

Green's function solution

$$T(r) = \int \frac{g(r')}{k} G(r \mid r')\, dv' \quad \text{(for volume energy generation)}$$

$$+ \sum_j \int_{s_j} \frac{f_j}{k} G(r, \mid r'_j) ds'_j \quad \text{(for b. c. of type 2 and 3)}$$

$$- \sum_i \int_{s_i} f_i \frac{\partial G(r \mid r'_i)}{\partial n'_i} ds'_i \quad \text{(for b. c. of type 1 only)}$$

Green's function G is the response at location r to an infinitessimal heat source located at coordinate r'.

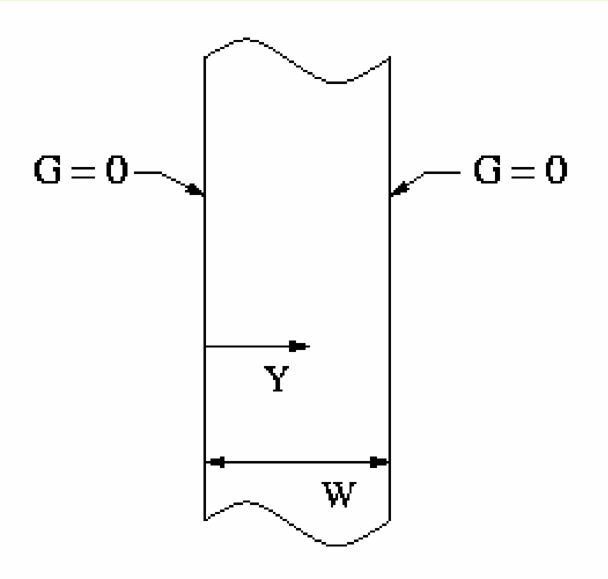
Green's function for 1D Slab

$$\frac{d^2G}{dy^2} = -\delta(y - y'); \quad 0 < y < W$$

$$k_i\frac{dG}{dn_i} + h_iG = 0; \quad i = 1 \text{ or } 2$$

Boundary conditions are homogeneous, and of the same *type* (1, 2, or 3) as the temperature problem. There are $3^2 = 9$ combinations of boundary types for the 1D slab.

1D Example



G=0 at y=0 and at y=W.

Y11 case. Two forms:

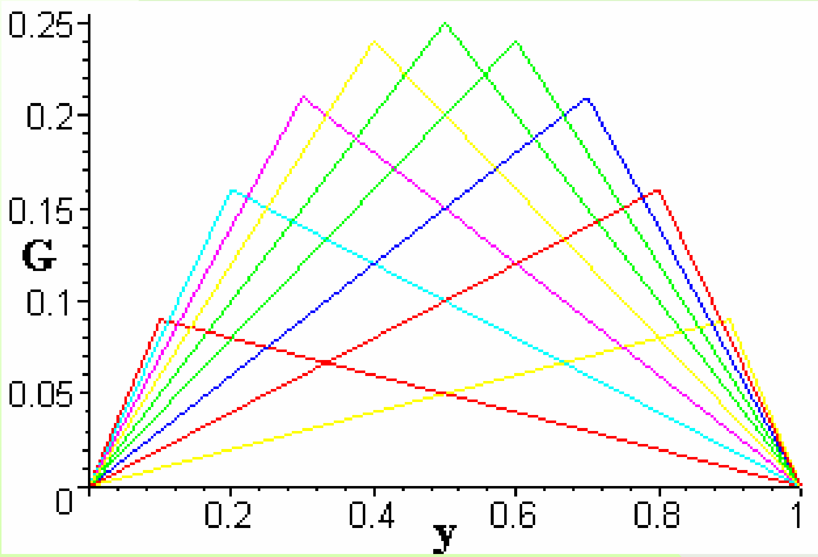
Series.

$$G(y,y') = \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{\sin(\gamma_n y)\sin(\gamma_n y')}{W/2}$$

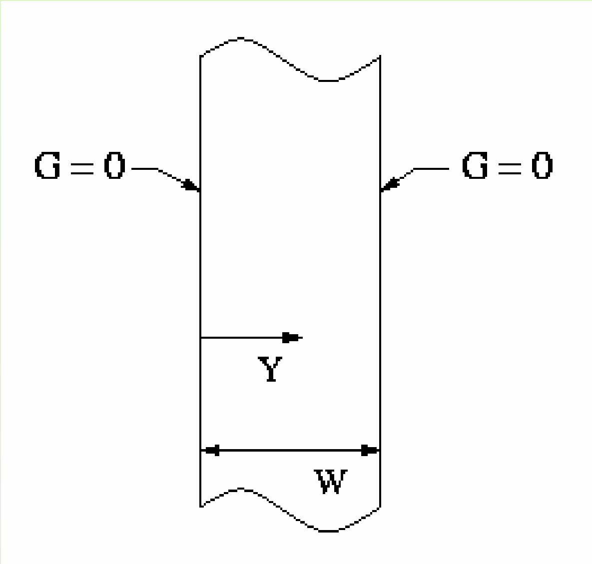
Polynomial.

$$G(y,y') = \begin{cases} y(1-y'/W); & y < y' \\ y'(1-y/W); & y > y' \end{cases}$$

Y11 case, continued



Plot of $G(y,y')$ versus y for several y' values.



Y11 Geometry.

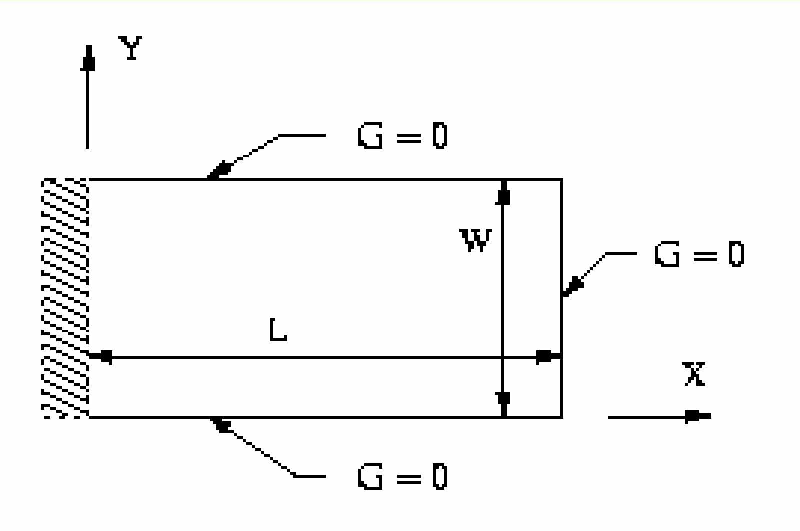
GF for the 2D Rectangle

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x - x')\delta(y - y'); \quad 0 < x < L; \; 0 < y < W$$

$$k_i \frac{\partial G}{\partial n_i} + h_i G = 0 \quad \text{for faces } i = 1, 2, 3, 4$$

- Here G is dimensionless.
- There are $3^4 = 81$ different combinations of boundary conditions (different GF) in the rectangle.

2D Example



Case X21Y11. *G*=0 at edges, except insulated at *x*=0.

Double sum form:

$$G(x, y \mid x', y') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos(\lambda_m x')\cos(\lambda_m x)}{(L/2)(W/2)} \frac{\sin(\gamma_n y)\sin(\gamma_n y')}{(\gamma_n^2 + \lambda_m^2)}$$

where $\gamma_n = n\pi/W$ $\lambda_m = (m - 1/2)\pi/L$

2D Example, case X21Y11

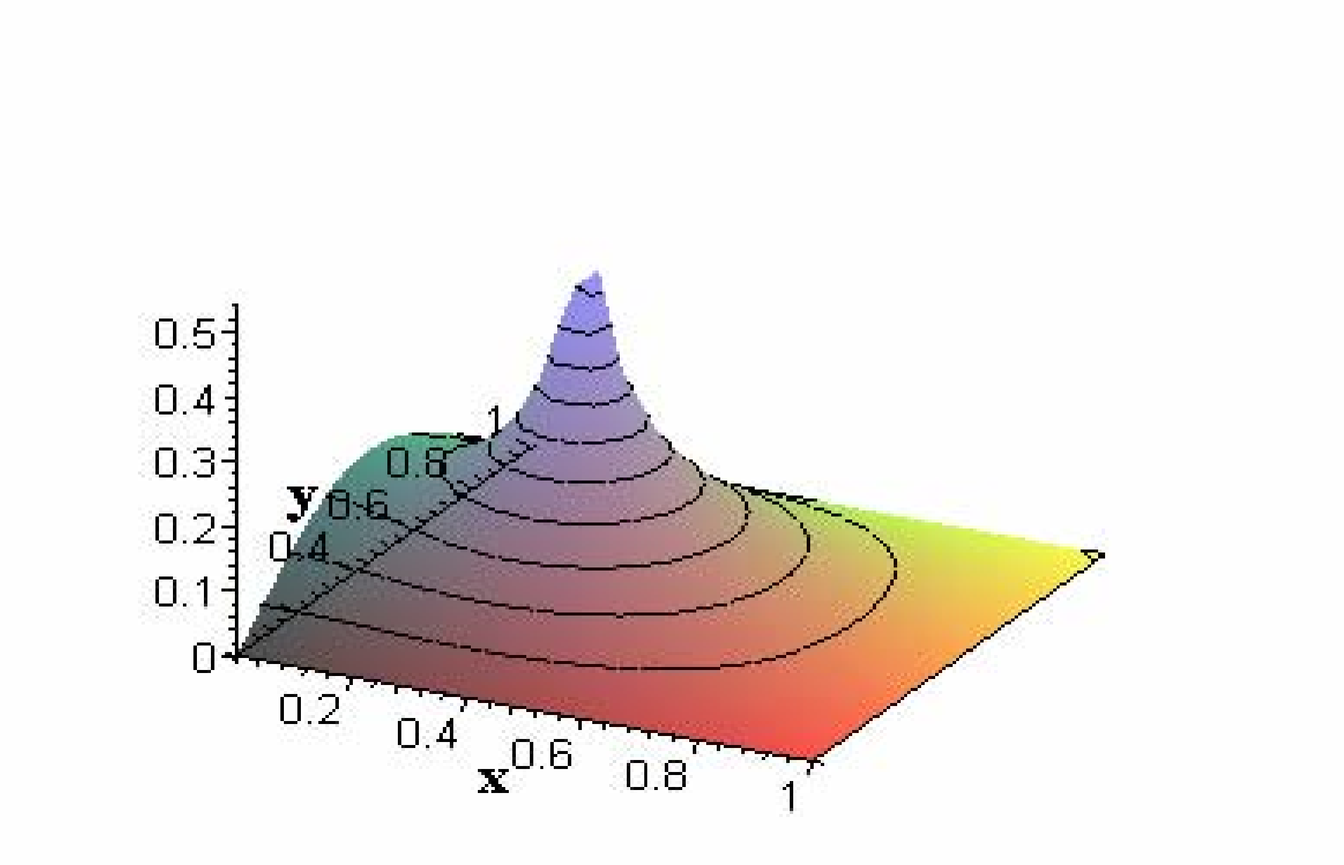
Single sum form:

$$G(x,y\,|\,x',y') = \sum_{n=1}^{\infty} \frac{\sin(\gamma_n y)\sin(\gamma_n y')}{W/2} P_n(x,x')$$

where kernel function P_n for this case is:

$$\begin{aligned} P_n(x,x') \quad = \quad & \{-\exp[-\gamma_n(2L-|x-x'|)] - \exp[-\gamma_n(2L-x-x')] \\ & +\exp[-\gamma_n|x-x'|] + \exp[-\gamma_n(x+x')]\} \\ & \div\{2\gamma_n[1+\exp(-2\gamma_n L)]\} \end{aligned}$$

Case X21Y11 heated at (0.4,0.4)

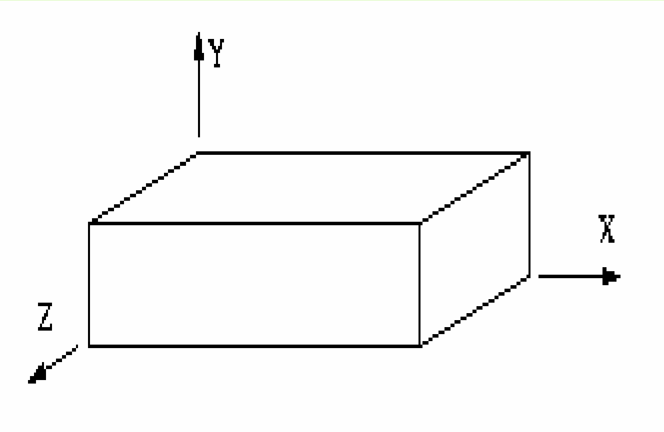


GF for the 3D Parallelepiped

$$
\begin{aligned}
\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} &= -\delta(x - x')\delta(y - y')\delta(z - z') \\
& \quad 0 < x < L;\ 0 < y < W;\ 0 < z < H \\
k_i \frac{\partial G}{\partial n_i} + h_i G &= 0 \ \text{ for faces } i = 1, 2, \ldots, 6
\end{aligned}
$$

There are 3^6=729 combinations of boundary types.

3D Example
Case X21Y11Z12



Triple sum form:

$$G = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\cos(\lambda_m x)\cos(\lambda_m x')\sin(\gamma_n y)\sin(\gamma_n y')}{(L/2)(W/2)(H/2)} \times \frac{\sin(\eta_p z)\sin(\eta_p z')}{(\lambda_m^2 + \gamma_n^2 + \eta_p^2)}$$

3D Example, X21Y11Z12 Alternate double-sum forms:

$$G = \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(\gamma_n y')\sin(\gamma_n y)}{(W/2)(H/2)} \frac{\sin(\eta_p z)\sin(\eta_p z')}{(\gamma_n^2+\eta_p^2)} P_{np}(x,x')$$

$$G = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{\cos(\lambda_m x')\cos(\lambda_m x)}{(L/2)(H/2)} \frac{\sin(\eta_p z)\sin(\eta_p z')}{(\lambda_m^2+\eta_p^2)} P_{mp}(y,y')$$

$$G = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\cos(\lambda_m x')\cos(\lambda_m x)}{(L/2)(W/2)} \frac{\sin(\gamma_n y)\sin(\gamma_n y')}{(\lambda_m^2+\gamma_n^2)} P_{mn}(z,z')$$

Web Publication: Promise

- Material can be presented in multiple digital formats, may be cut and pasted into other digital documents.
- Immediate world-wide distribution.
- Retain control of content, easily updated.
- Hyperlinks to related sites.

Web Publication: Pitfalls

- No editorial support, no royalties.
- Unclear copyright protection.
- Continued operating costs (service provider, computer maintenance, etc.)
- Little academic reward; doesn't "count" as a publication.

NIST Digital Library of Mathematical Functions

- Web-based revision of handbook by Abramowitz and Stegun (1964).
- Emphasis on text, graphics with few colors, photos used sparingly.
- Navigational tools on every page.
- No proprietary file formats (HTML only).
- Source code developed in AMS-TeX.

Green's Function Library

- Source code is LateX, converted to HTML with shareware code latex2html run on a Linux PC
- GF are organized by equation, coordinate system, body shape, and type of boundary conditions
- Each GF also has an identifying number

Contents of the GF Library

- **Heat Equation. Transient Heat Conduction**

    Rectangular Coordinates. Transient 1-D

    Cylindrical Coordinates. Transient 1-D

    Radial-Spherical Coordinates.Transient 1-D
- **Laplace Equation. Steady Heat Conduction**

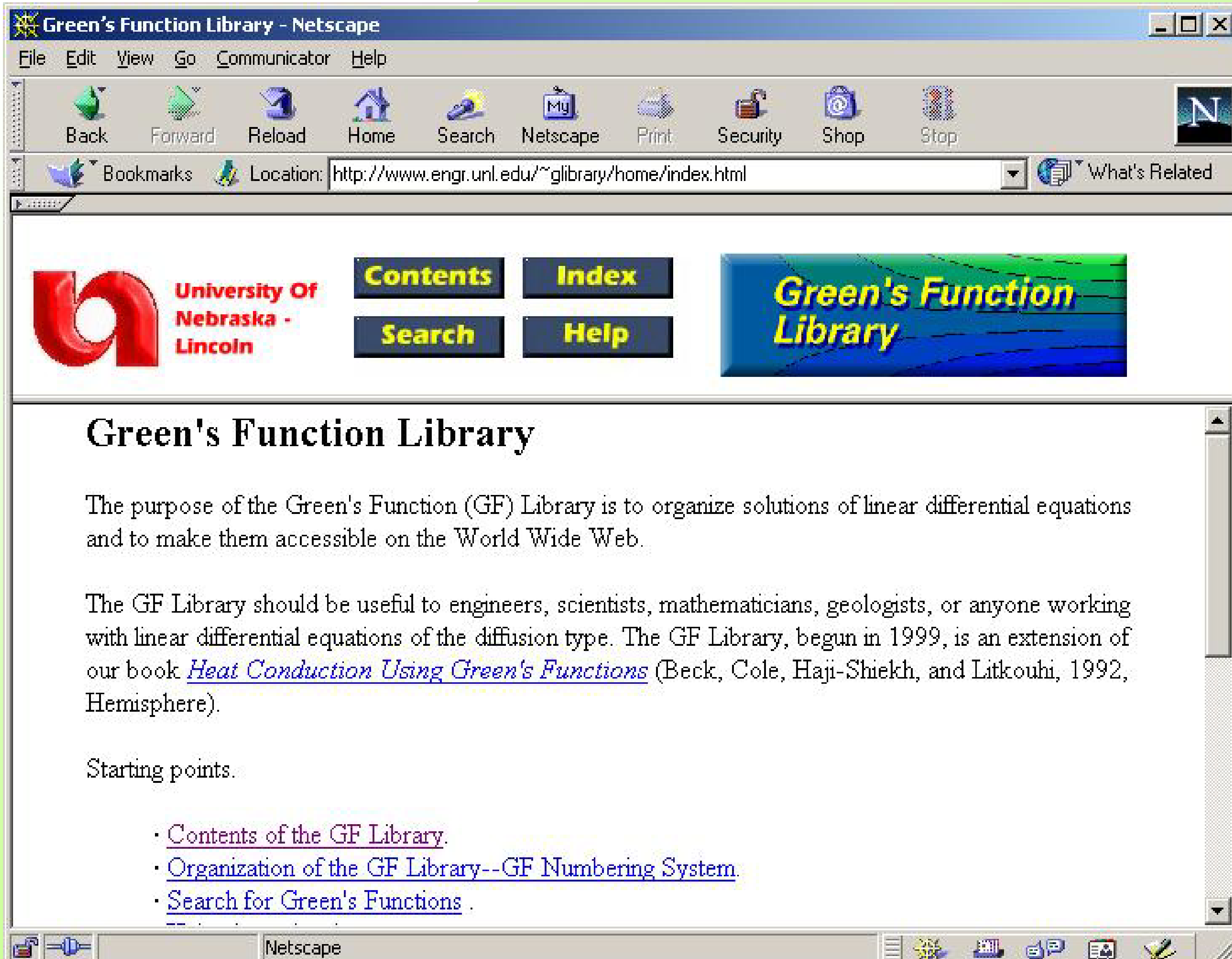
    Rectangular Coordinates. Steady 1-D

    Rectangular Coordinates. Finite Bodies, Steady.

    Cylindrical Coordinates. Steady 1-D

    Radial-Spherical Coordinates.Steady 1-D
- **Helmholtz Equation. Steady with Side Losses**

    Rectangular Coordinates. Steady 1-D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Green's Function Library
The purpose of the Green's Function (GF) Library is to organize solutions of linear differential equations and to make them accessible on the World Wide Web.
The GF Library should be useful to engineers, scientists, mathematicians, geologists, or anyone working with linear differential equations of the diffusion type. The GF Library, begun in 1999, is an extension of our book Heat Conduction Using Green's Functions (Beck, Cole, Haji-Shiekh, and Litkouhi, 1992, Hemisphere).
Starting points.
· Contents of the GF Library.
· Organization of the GF Library--GF Numbering System.
· Search for Green's Functions .
Netscape

Plate, steady 1-D.

X11 Plate, G=0 (Dirichlet) at x=0 and x=L.

$$G_{X11}(x\,|x') = \begin{cases} x(1-x')/L & \text{for } x < x' \\ x'(1-x)/L & \text{for } x > x' \end{cases}$$

X12 Plate, G=0 (Dirichlet) at x=0 and $\partial G/\partial x = 0$ (Neumann) at x=L.

$$G_{X12}(x\,|x') = \begin{cases} x & \text{for } x < x' \\ x' & \text{for } x > x' \end{cases}$$

X13 Plate, G=0 (Dirichlet) at x=0 and $k\partial G/\partial x + h_2 G = 0$ (Neumann) at x=L Note B_2=h_2L/k

$$G_{X13}(x\,|x') = \begin{cases} x[1-B_2(x'/L)/(1+B_2)] & \text{for } x < x' \\ x'[1-B_2(x/L)/(1+B_2)] & \text{for } x > x' \end{cases}$$

Solid cylinder transient 1-D. - Netscape

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Solid cylinder transient 1-D.

R01 Solid cylinder $0<r<b$, with $G=0$ (Dirichlet) at $r=b$.

$$G_{R01}(r,t\mid r',\tau) = \frac{1}{\pi b^2}\sum_{m=1}^{\infty}\exp\left[-\beta_m^2\alpha(t-\tau)/b^2\right] \times \frac{J_0(\beta_m r/b)\, J_0(\beta_m r'/b)}{\left[J_1(\beta_m)\right]^2}$$

with eigenvalues given by $J_0(\beta_m)=0$.

Document: Done

Summary

- GF in slabs, rectangle, and parallelepiped for 3 types of boundary conditions
- These GF have components in common: 9 eigenfunctions and 18 kernel functions
- Alternate forms of each GF allow efficient numerical evaluation

Summary, continued.

Web Publishing: wide dissemination, local control, updatable; continuing expense, little academic reward.

Green’s Function Library: source code developed in LateX (runs on any computer) and converted to HTML with latex2html (runs on Linux).

Work in progress: Dynamic Math

- Currently GF web page is static, book-like
- Temperature solutions are too numerous for pre-determined display
- Working to create and display temperature solutions on demand, in response to user input.
- Code with open standards Perl, latex2html

Acknowledgments

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