Direct Evaluation of 3D Hypersingular Galerkin Integrals

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Outline

- **Direct Evaluation:**
  - definition as limit to the boundary – finite integrals
  - polar coordinates: $\rho\, d\rho$ reduces singularity and ρ integrated analytically
  - *explicit* identification and cancellation of divergences for coincident & edge-adjacent integrals
- **Surface gradient evaluation**

Hypersingular Integral: Laplace

Collocation:

$$\int_\Sigma \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P_0, Q)\, \mathrm{d}Q$$

Galerkin:

$$\int_\Sigma \psi_k(P) \int_\Sigma \phi(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q)\, \mathrm{d}Q\, \mathrm{d}P$$

$$\frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q) = \frac{1}{4\pi} \left(\frac{\mathbf{n} \bullet \mathbf{N}}{r^3} - 3 \frac{(\mathbf{n} \bullet R)(\mathbf{N} \bullet R)}{r^5} \right)$$

$$\mathbf{R} = Q - P \qquad r = \| \mathbf{R} \|$$

$$r = 0$$

Galerkin: Divergent Integrals

Interpolation: $\phi(Q) = \sum_j \phi(Q_j)\psi_j(Q)$

$$\sum_j \phi(P_j) \int_{E_P} \psi_k(P) \int_{E_Q} \psi_j(Q) \frac{\partial^2 G}{\partial \mathbf{N} \partial \mathbf{n}}(P, Q)\, \mathrm{d}Q\, \mathrm{d}P$$

An individual integral is **divergent** if:

- $E_P = E_Q$
- E_P and E_Q share an edge

Root cause is same as $\mathcal{C}^1$, must integrate over complete neighborhood of the singularity.

Galerkin Evaluation in 3D

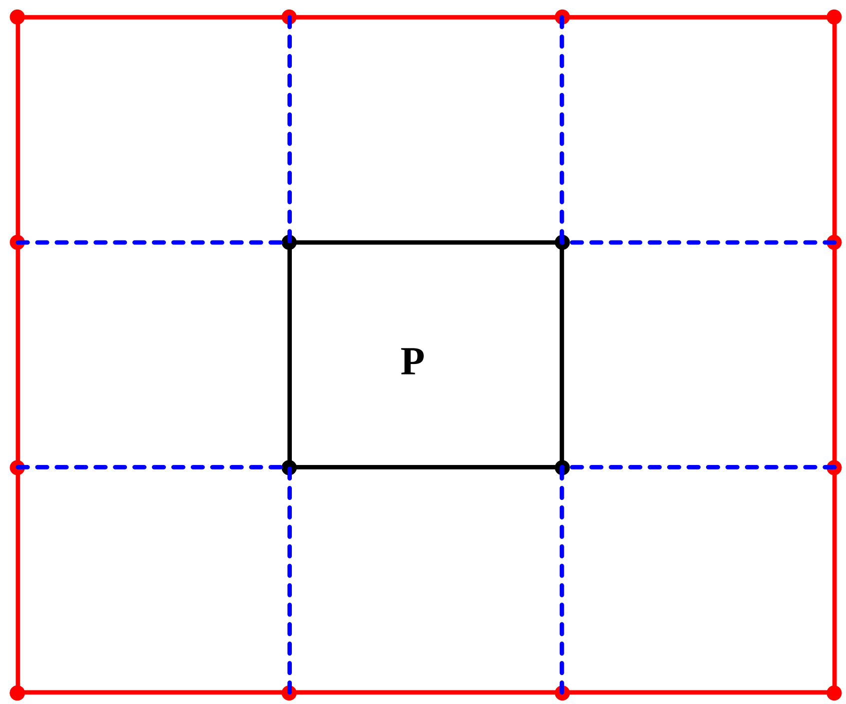
Stokes' Theorem: (Li & Mear, Frangi & Novati, Bonnet, Lutz)

- finite integrals
- contour integral
- 'smearing' of local contributions, generality(?) (*e.g.*, FGM Green's functions)

Hadamard Finite Part: (Salvadori & Carini)

- analytic integration (for inner integral)
- artificially removes singularity
- complicated, generality(?)

Stokes Contour



Limit Direct

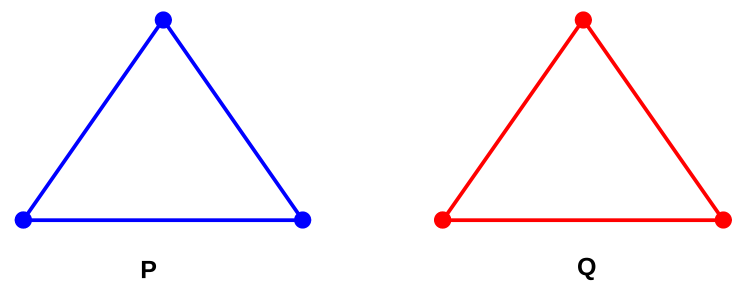
Goals:

- Materialize divergent terms
- Prove cancellation
- Generally applicable algorithm

Method:

- **Boundary Limit:** $P \to P + \epsilon \mathbf{N}$
- Change variables, swap integrations, analytic ρ integration
- Cancellation via complete integration

Parameter Spaces



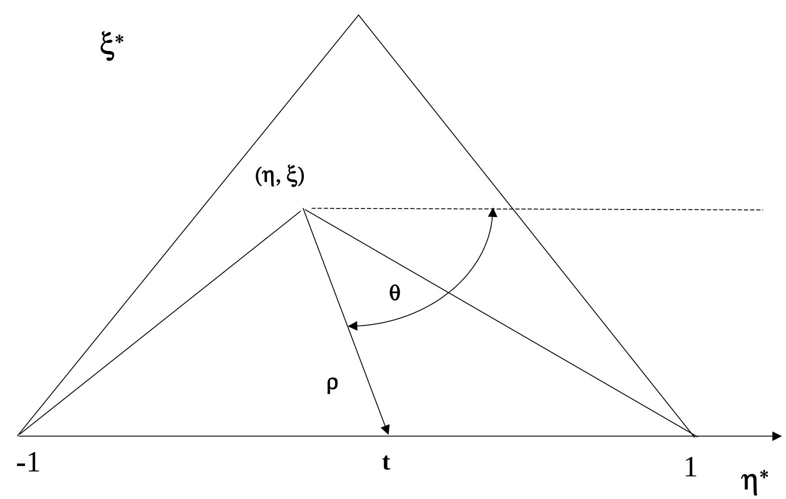
Equilateral triangle: $\mathrm{d}P \rightarrow \mathrm{d}\xi\,\mathrm{d}\eta \qquad \mathrm{d}Q \rightarrow \mathrm{d}\xi^*\,\mathrm{d}\eta^*$

$$-1 \leq \eta \leq 1 \qquad 0 \leq \xi \leq \sqrt{3}\,(1-|\eta|)$$

Linear Element:

$$\psi_1(\eta,\xi) = (1-\eta-\xi/\sqrt{3})/2$$
$$\psi_2(\eta,\xi) = (1+\eta-\xi/\sqrt{3})/2$$
$$\psi_3(\eta,\xi) = \xi/\sqrt{3}$$

Coincident: First Integration

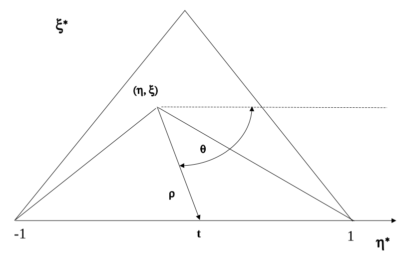


$$r^2 = a^2 \rho^2 + \epsilon^2$$

$$\int_{-1}^{1} \mathrm{d}\eta \int_{0}^{\sqrt{3}(1-|\eta|)} \mathrm{d}\xi \int_{\theta_1(\eta,\xi)}^{\theta_2(\eta,\xi)} \frac{\rho_L^2}{(\epsilon^2 + a^2 \rho_L^2)^{3/2}} \, \mathrm{d}\theta$$

$$\rho_L \to 0 \text{ for } \xi \to 0$$

Coincident: continued



$$\int_{-1}^{1} \mathrm{d}\eta \int_{0}^{\sqrt{3}(1-|\eta|)} \mathrm{d}\xi \int_{\theta_1}^{\theta_2} \frac{\rho_L^2}{(\epsilon^2 + a^2\rho_L^2)^{3/2}} \, \mathrm{d}\theta$$

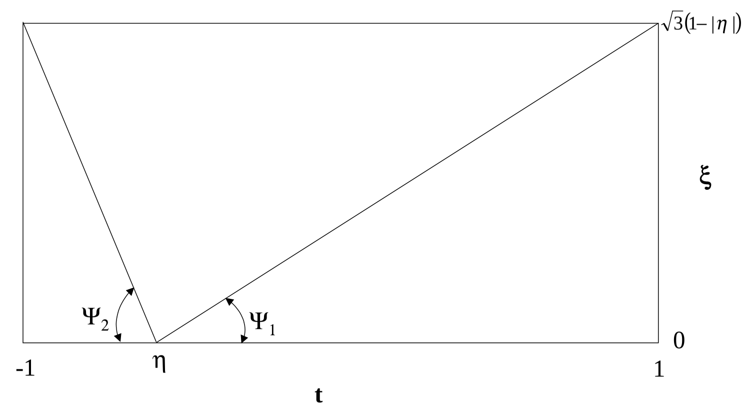
Change of Variables: $\theta = -\frac{\pi}{2} + \tan^{-1}(\frac{t-\eta}{\xi}) \qquad -1 \leq t \leq 1$

$$\int_{-1}^{1} \mathrm{d}\eta \int_{-1}^{1} \mathrm{d}t \int_{0}^{\sqrt{3}(1-|\eta|)} \frac{\cdots}{[\epsilon^2 + a^2(\xi^2 + (t-\eta)^2)]^{3/2}} \, \mathrm{d}\xi$$

Coincident: Second Integration

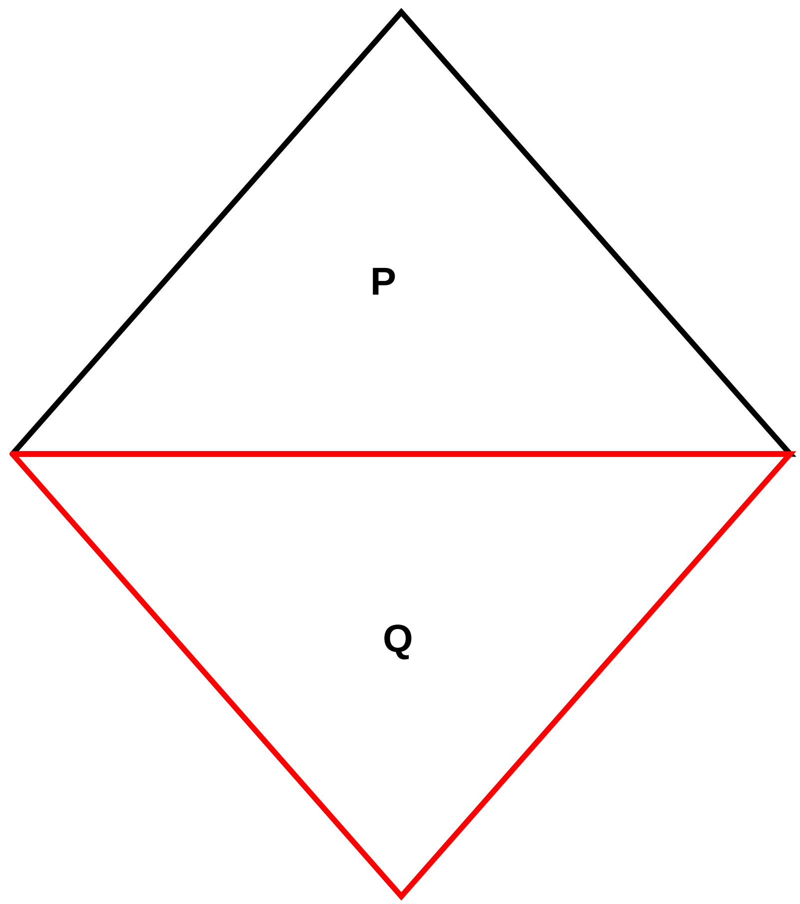
Denominator suggests another polar transformation for (t, ξ)

$$t - \eta = \Lambda \cos(\Psi) \qquad \xi = \Lambda \sin(\Psi)$$

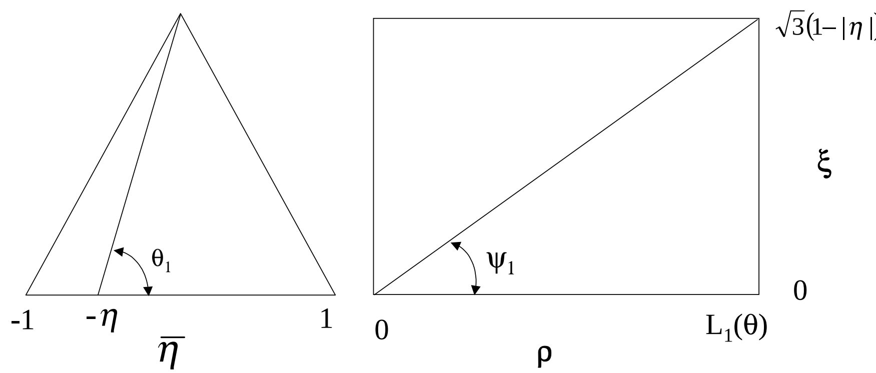


$$\int \frac{\Lambda^2}{(\epsilon^2 + a^2\Lambda^2)^{3/2}}\, \mathrm{d}\Lambda \rightarrow \log(\epsilon)$$

Adjacent Edge



Singularity at: $\xi = \xi^* = 0 \qquad \eta = -\eta^*$



(a) (b)

1. Polar coordinates $(\rho,\ \theta)$ centered at $(-\eta,\ 0)$ replacing $(\eta^*,\ \xi^*)$
2. θ_1 only a function of $\eta \Rightarrow$ interchange ξ and θ
3. Polar coordinates $(\Lambda,\ \Psi)$ replacing (ρ, ξ); (η, θ, Ψ) numerically
4. Jacobian $= \Lambda^2 \cos(\Psi)$; $r^2 = \epsilon^2 + a_1\epsilon\Lambda + a_2\Lambda^2$

Comments

- Analytic integration is always wrt distance from the singularity
- Equilateral parameter space for convenience – coincident integration
- Proving cancellation of the $\log(\epsilon)$ terms takes some work – remaining dimensions must be integrated
- **Adjacent Edge**: limit direction not necessarily normal
- Most of the grunge work handled by symbolic computation

Limit vs. Stokes

Simplicity is in the eye of the beholder

- Gradient evaluation
- FGM Green's Functions
- Conditioning (?): nonlinear boundary conditions (electrochemistry, crack friction, etc.)

Gradient applications:

- Moving boundary problems: wave motion, electromigration, crystallization, etc.
- Nonlinear: Contact analysis, Shape optimization

Galerkin Gradient I

$$\int_\Sigma \hat{\psi}_k(P)\frac{\partial\phi}{\partial\mathbf{E}_k}(P)\ \mathrm{d}P = \lim_{P_I\to P}\int_\Sigma \hat{\psi}_k(P) \int_\Sigma \left[\frac{\partial G}{\partial\mathbf{E}_k}(P_I,Q)\frac{\partial\phi}{\partial\mathbf{n}}(Q) - \phi(Q)\frac{\partial^2 G}{\partial\mathbf{E}_k\partial\mathbf{n}}(P_I,Q)\right]\ \mathrm{d}Q\,\mathrm{d}P$$

$$A\left[\frac{\partial\phi(P_j)}{\partial\mathbf{E}_k}\right] = b$$

A is sparse, symmetric positive definite

b is an expensive computation

Galerkin Gradient II

Exterior Limit Equation:

$$0 = \lim_{P_E \to P} \int_\Sigma \hat{\psi}_k \int_\Sigma \left[\frac{\partial G}{\partial \mathbf{E}_k}(P_E, Q) \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial^2 G}{\partial \mathbf{E}_k \partial \mathbf{n}}(P_E, Q) \right] \mathrm{d}Q \, \mathrm{d}P$$

$$\frac{\partial \phi(P)}{\partial \mathbf{E}_k} = \lim_{P_I \to P} - \lim_{P_E \to P} \int_\Sigma \hat{\psi}_k \int_\Sigma \left[\frac{\partial G}{\partial \mathbf{E}_k} \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial^2 G}{\partial \mathbf{E}_k \partial \mathbf{n}} \right] \mathrm{d}Q \, \mathrm{d}P$$

1. *Only terms that are discontinuous crossing boundary*: coincident, hypersingular adjacent edge (low order terms from shape functions)
2. Requires a limit definition
3. Collocate limit difference?

Penny-shaped crack

Disk:

$$x^2 + y^2 \le R_0^2 = 0.02 \qquad z = 0$$

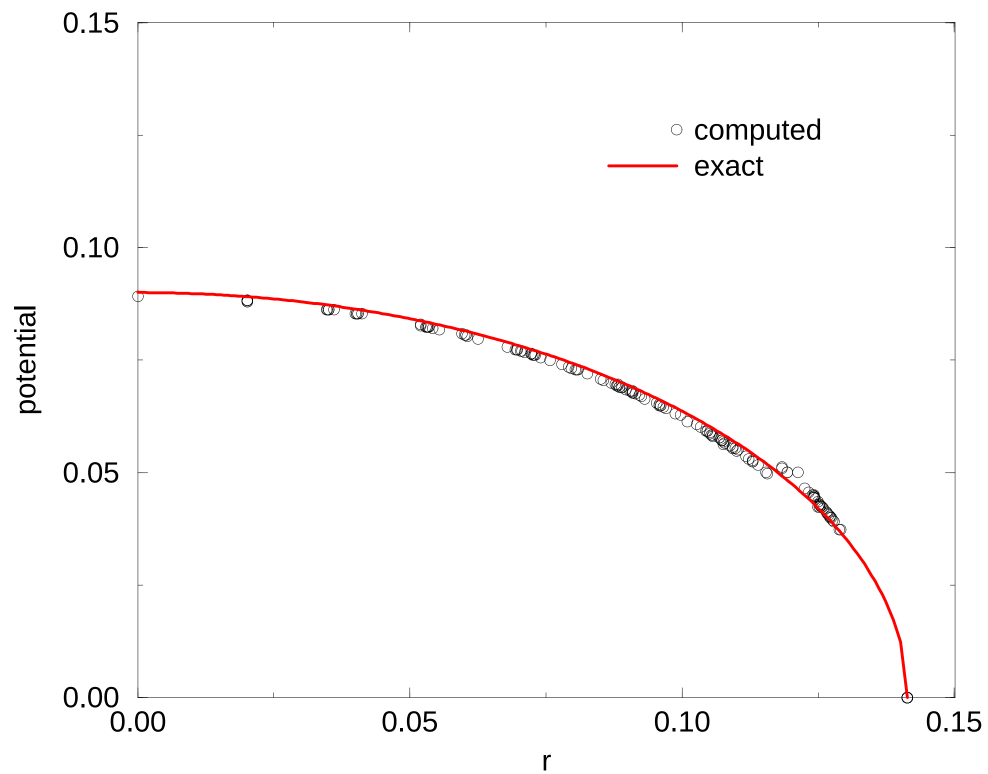
Boundary condition:

$$\left[\frac{\partial \phi}{\partial \mathbf{n}}\right] = \frac{\partial \phi}{\partial \mathbf{n}}(P^+) + \frac{\partial \phi}{\partial \mathbf{n}}(P^-) = 1$$

Solution:

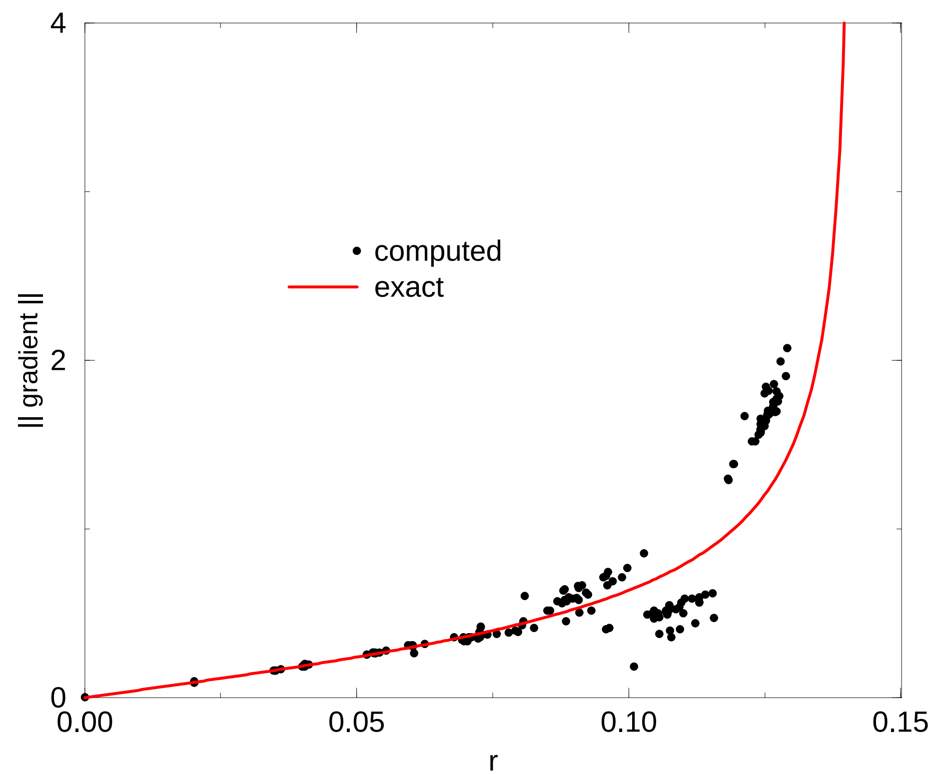
$$[\phi] = \phi(P^+) - \phi(P^-) = \frac{2R_0}{\pi} \sqrt{1 - \frac{r^2}{R_0^2}}$$

Penny-shaped crack



The solution for $[\,\phi\,]$ for the pressurized penny shaped crack.

Penny-shaped crack



Current Work

- Higher order elements: linear analysis plus remainders

$$\left(a_4\rho^4 + a_3\rho^3 + a_2\rho^2 + \epsilon^2\right)^{-1/2} = \left(a_2\rho^2 + \epsilon^2\right)^{-1/2} + \mathrm{Rem}$$

- Green's functions: elasticity (Phan), FGM (Sutradhar), anisotropic elasticity
- Conditioning
- Numerical integrations

Anisotropic Elasticity

$$U_{kj}(Q,P) = \frac{1}{8\pi^2 r}\tilde{U}_{kj}(\theta,\psi)$$

$$U_{kj,lm}(Q,P) = \frac{1}{8\pi^2} \left[\left(\frac{\delta_{lm}}{r^3} - 3\frac{(q_l - p_l)(q_m - p_m)}{r^5} \right) \tilde{U}_{kj}(\theta(\epsilon),\psi(\epsilon)) + \cdots \right]$$

Taylor expansion for $\tilde{U}_{kj}$

Summary

- Direct, general, simple, 3D singular integration algorithms
- Limit definition essential for fast surface gradient method

Collaborators

- Julia Glaeser (summer undergrad)
- Anh-Vu Phan (ORNL)
- Ted Kaplan (ORNL)